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Nonconvex vector optimization of set-valued mappings ☆

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Abstract

In this paper, we discuss properties, such as monotonicity and continuity, of the Gerstewitz's nonconvex separation functional. With the aid of this functional, necessary and sufficient optimality conditions for nonconvex optimization problems of set-valued mappings are obtained in topological vector spaces.
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1. Introduction

In the last decade, there has been an increasing interest in set-valued optimization (Refs. [1–6] and references therein). Optimization problems with set-valued constraints or set-valued objective functions are clearly related to problems in stochastic programming, fuzzy programming and optimal control. If the values of a given function vary in a specified region, this fact could be described using a membership function in the theory of fuzzy sets or using information on the distributions of the function value. In this general

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setting, probability distributions or membership functions are not needed because only set is considered. Optimal control problems with differential inclusions belong to this class of set-valued optimization problems as well. Set-valued optimization is a substantial extension of single-valued optimization theory.

Optimality conditions of solutions for set-valued optimization problems have been obtained by using contingent derivative, contingent epiderivative and generalized contingent epiderivative of set-valued mappings, respectively (Refs. [1–3]). In fact, these optimality conditions are in the form of vector variational inequalities.

In this paper, we aim to derive optimality conditions of solutions for set-valued optimization problems by using a nonconvex separation function (Ref. [7]). These optimality conditions are in the form of minimax. From obtained results, we show that cone-efficiency in vector-valued and set-valued optimization problems can reduce to Pareto-efficiency.

The paper is organized as follows. In Section 2, we introduce some notations and preliminary results. Based on these preliminary discussions, we discuss properties of nonconvex separation functions. In Section 3, we get two optimality conditions of set-valued optimization problems. By examples, we show that general cone-efficiency can reduce to Pareto-efficiency.

2. Preliminary

Let X and V be two topological vector spaces and $S \subset V$ closed convex pointed cone with nonempty interior $\operatorname{int} S \neq \emptyset$. ∂S denotes the topological boundary of S. V^* denotes the topological dual space of V. Some fundamental terminologies and preliminary results are presented as follows.

Definition 2.1. Given fixed $k \in \operatorname{int} S$ and $a \in V$, the Gerstewitz's nonconvex separation functional $\xi_{ka} : V \to R$ is defined by

$$\xi_{ka}(y) = \min\{t \in R \mid y \in a + tk - S\} \quad \text{for } y \in V.$$

Definition 2.2. A function $\psi : V \to R$ is strictly monotone if

$$y_1 - y_2 \in \operatorname{int} S \Rightarrow \psi(y_1) > \psi(y_2).$$

Lemma 2.1 (See Theorem 2.1 of Ref. [7]). *For any fixed $k \in \operatorname{int} S$ and $y \in V$, we have*:

(i) $\xi_{ka}(y) < r \Leftrightarrow y \in a + rk - \operatorname{int} S$;
(ii) $\xi_{ka}(y) \leqslant r \Leftrightarrow y \in a + rk - S$;
(iii) $\xi_{ka}(y) = 0 \Leftrightarrow y = a - \partial S$;
(iv) $\xi_{ka}(y) > r \Leftrightarrow y \notin a + rk - S$;
(v) $\xi_{ka}(y) \geqslant r \Leftrightarrow y \notin a + rk - \operatorname{int} S$.

Let $C \subset V$. The generated cone of C is defined by

$$\operatorname{cone} C := \{tc \mid t \geqslant 0,\ c \in C\}.$$

Let $C \subset V$. The dual cone of C is defined as

$$C^+ := \{ f \in V^* \mid f(c) \geqslant 0, \text{ for any } c \in C \}.$$

Lemma 2.2. *$C \subset V$ is a closed convex cone if and only if there exists a $\Gamma \subset V^* \backslash \{0_V\}$ such that*

$$C = \{ y \in V \mid f(y) \geqslant 0, \textit{ for any } f \in \Gamma \}.$$

Proof. We take any $\bar{y} \notin C$. Then $\text{cone}(\bar{y})$ is pointed, closed and convex cone, and it obviously is locally compact (i.e., the set $\text{cone}(\bar{y})$ has a compact neighborhood base in the relative topology on $\text{cone}(\bar{y})$) and

$$\text{cone}(\bar{y}) \cap C = \{0_V\}.$$

Therefore, by Proposition 3 of Ref. [8] there exists a $-f_{\bar{y}} \in V^*$, such that

$$-f_{\bar{y}}(z) > 0, \quad \text{for any } z \in \text{cone}(\bar{y}) \backslash \{0_V\};$$
$$-f_{\bar{y}}(z) \leqslant 0, \quad \text{for any } z \in C.$$

Let $\Gamma = \{ f_{\bar{y}} \in V^* \mid \bar{y} \notin C\}$. Define

$$P := \{ y \in V \mid f(y) \geqslant 0, \ f \in \Gamma \}.$$

Now we shall prove that $C = P$.

In fact, let $y \in C$. By the construction of Γ, we have that

$$f(y) \geqslant 0, \quad \text{for any } f \in \Gamma.$$

Thus, $y \in P$.

Conversely, let $y \in P$ and $y \notin C$. There exists a $-f_y \in V^*$, such that

$$-f_y(y) > 0;$$
$$-f_y(z) \leqslant 0, \quad \text{for any } z \in C.$$

Obviously, $f_y \in \Gamma$, which contradicts $y \in P$. Thus, the conclusion holds. □

The following lemma is obviously.

Lemma 2.3. *Let $k \in \text{int}\, S$. Then, for any $f \in S^+ \backslash \{0_{V^*}\}$, $f(k) > 0$.*

Proposition 2.1. *Let $S \subset V$ be a closed convex cone with $\text{int}\, S \neq \emptyset$. Let $k \in \text{int}\, S$. Then there exists a $\Gamma \subset S^+ \backslash \{0_{V^*}\}$ such that*

$$S = \{ y \in V \mid f(y) \geqslant 0, \textit{ for any } f \in \Gamma \}$$

and

$$\xi_{ka}(y) = \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(k)} \right\}.$$

Proof. By Proposition 2.3 in Ref. [9] and Lemma 2.1, the conclusion holds. □

Let S be a closed convex cone with nonempty interior and let $k \in \operatorname{int} S$. We define

$$S_k^+ = \{f \in V^* \mid f(s) \geqslant 0,\ \text{for } s \in S,\ f(k) = 1\}.$$

Corollary 2.1. *Let $S \subset V$ be a closed convex cone with $\operatorname{int} S \neq \emptyset$ and let $k \in \operatorname{int} S$. Then, there exists a $\Gamma \subset S_k^+$ such that*

$$S = \{y \in V \mid f(y) \geqslant 0,\ \textit{for any } f \in \Gamma\}$$

and

$$\xi_{ka}(y) = \sup_{f \in \Gamma}\{f(y) - f(a)\}.$$

Proof. By Proposition 2.1, there exists a $\Gamma \subset S^+ \backslash \{0_{V^*}\}$ such that

$$S = \{y \in V \mid f(y) \geqslant 0,\ \text{for any } f \in \Gamma\}$$

and

$$\xi_{ka}(y) = \sup_{f \in \Gamma}\left\{\frac{f(y) - f(a)}{f(k)}\right\}.$$

Since $k \in \operatorname{int} S$, it follows from Lemma 2.3 that $f(k) > 0$, for $f \neq 0_{V^*}$.

Set

$$\overline{\Gamma} = \left\{f' \;\middle|\; f' = \frac{f}{f(k)},\ \text{for any } f \in \Gamma\right\}.$$

Obviously, for any $f' \in \overline{\Gamma}$, $f'(k) = 1$. Therefore,

$$\overline{\Gamma} \subset S_k^+.$$

Thus, it is clear that

$$S = \{y \in V \mid f(y) \geqslant 0,\ \text{for any } f \in \overline{\Gamma}\} \quad \text{and} \quad \xi_{ka}(y) = \sup_{f' \in \overline{\Gamma}}\{f'(y) - f'(a)\},$$

and the proof is complete. □

Example 2.1. Let $V = R^2$ and let S be a closed convex cone (see Fig. 1). Suppose that $\Gamma = \{f_1, f_2\}$, where

$$f_1(z) = -(-3, 1)\begin{pmatrix} x \\ y \end{pmatrix}, \quad f_2(z) = -(1, -3)\begin{pmatrix} x \\ y \end{pmatrix}, \quad z = (x, y).$$

Then, by Fig. 1 of S, we have

$$\begin{aligned} S &:= \{z \in V \mid f(z) \geqslant 0,\ \text{for any } f \in \Gamma\} \\ &= \left\{(x, y) \in V \;\middle|\; f_1(z) = (-3, 1)\begin{pmatrix} x \\ y \end{pmatrix} \geqslant 0,\ f_2(z) = (1, -3)\begin{pmatrix} x \\ y \end{pmatrix} \geqslant 0\right\}. \end{aligned}$$

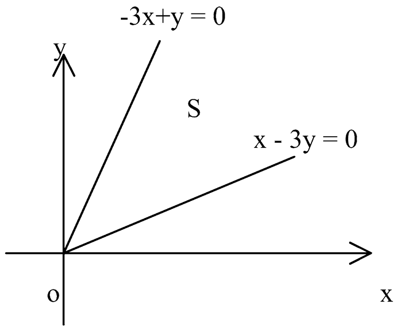

Fig. 1. The set S.

Take $k = (1, 1) \in \operatorname{int} S$. Then,

$$\xi_{k0}(x, y) = \begin{cases} \frac{1}{2}(3x - y), & x \geqslant y, \\ \frac{1}{2}(3y - x), & y \geqslant x. \end{cases}$$

3. Optimality conditions for nonconvex set-valued mappings

In this section, we consider optimality conditions for constrained set-valued optimization problems. Let X, V and Z be three topological vector spaces. V^* and Z^* denote the topological dual spaces of V and Z, respectively. Let $S \subset V$ and $P \subset Z$ be two closed convex cones with nonempty interiors, and let $k \in \operatorname{int} S$ and $e \in \operatorname{int} P$. Let $F : D \subset X \to 2^V$ and $G : D \subset X \to 2^Z$ be two set-valued mappings.

Consider generalized vector program:

$$\text{(GVP)} \quad \min F(x), \\ \text{s.t.} \quad -G(x) \cap P \neq \emptyset.$$

We denote by K feasible set of (P), namely

$$K = \{x \in D \mid -G(x) \cap P \neq \emptyset\}.$$

Definition 3.1. Let $x_0 \in K$ and $y_0 \in F(x_0)$. A pair (x_0, y_0) is said to a minimal solution of (GVP), if for any $x \in K$, there exists no $y \in F(x)$ such that

$$y \in y_0 - S \backslash \{0_V\},$$

namely

$$\big(F(K) - y_0\big) \cap (-S) = \{0_V\}.$$

Lemma 3.1. (1) $h \in (V \times Z)^*$ *if and only if there exist* $f \in V^*$ *and* $g \in Z^*$ *such that*

$$h(x, z) = f(x) + g(z), \quad \textit{for any } (x, z) \in V \times Z.$$

(2) $h \in (S \times P)^+$ *if and only if there exist* $f \in S^+$ *and* $g \in P^+$ *such that*

$$h(x, z) = f(x) + g(z), \quad \textit{for any } (x, z) \in V \times Z.$$

Proof. Since h is a linear function, $h(x,z) = h(x,0) + h(0,z)$. Set $f(x) = h(x,0)$ and $g(z) = h(0,z)$. Thus, f and g satisfy the conditions of lemma and the proof is complete. □

Lemma 3.2. *Let $L \subset V^*$ and $\Gamma \subset Z^*$. Suppose that*

$$S := \{x \in V \mid f(x) \geqslant 0,\ \text{for any } f \in L\}$$

and

$$P := \{z \in V \mid g(z) \geqslant 0,\ \text{for any } g \in \Gamma\}.$$

Then,

$$S \times P = \{(x,z) \mid f(x) + g(z) \geqslant 0,\ \text{for any } f \in L \cup \{0_{V^*}\},\ g \in \Gamma \cup \{0_{Z^*}\} \\ \text{and } (f,g) \neq (0_{V^*}, 0_{Z^*})\}.$$

Proof. By Lemma 3.1, the conclusion follows readily. □

Theorem 3.1. *Let $x_0 \in K$ and $y_0 \in F(x_0)$. Suppose that (x_0, y_0) is a minimal solution of* (GVP). *Then, for any $L \subset S^+$ and $\Gamma \subset P^+$, which satisfy*

$$f(k) + g(k) = 1, \quad \text{for } f \in L,\ g \in \Gamma \text{ and } (f,g) \neq (0_{V^*}, 0_{Z^*}), \tag{1}$$

$$S := \{x \in V \mid f(x) \geqslant 0,\ \text{for any } f \in L\}, \tag{2}$$

and

$$P := \{z \in V \mid g(z) \geqslant 0,\ \text{for any } g \in \Gamma\}, \tag{3}$$

we have

$$\inf\left(\bigcup_{x \in K}\left(\bigcup_{y \in F(x)} \sup_{f \in L}\{f(y) - f(y_0)\} + \bigcup_{z \in G(x)} \sup_{g \in \Gamma}\{g(z)\}\right)\right) = 0 \tag{4}$$

and

$$\inf\left(\bigcup_{z \in G(x_0)} \sup_{g \in \Gamma}\{g(z)\}\right) = 0.$$

Proof. By Definition 3.1, we get

$$\big(F(K) - y_0\big) \cap (-S) = \{0_V\}.$$

Namely,

$$\big[\big(F(K)\backslash\{y_0\}\big) - y_0\big] \cap (-S) = \emptyset.$$

Thus, for any $x \in K$, we have

$$\big[\big(F(x)\backslash\{y_0\}\big) - y_0\big] \times G(x) \cap (-S \times P) = \emptyset. \tag{5}$$

For any $L \subset S^+$ and $\Gamma \subset P^+$, which satisfy (1), (2) and (3), it follows from Lemma 3.2 that

$$S \times P = \bigl\{(y,z) \in V \times Z \mid f(y) + g(z) \geqslant 0, \text{ for any } f \in L \cup \{0_{V^*}\},\ g \in \Gamma \cup \{0_{Z^*}\} \text{ and } (f,g) \neq (0_{V^*}, 0_{Z^*})\bigr\}.$$

Thus, by (5), Proposition 2.3 of Ref. [9] and Lemma 2.1(iv), it is clear that, for any $x \in K$,

$$\sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}} \left\{\frac{f(y)+g(z)}{f(k)+g(k)}\right\} > 0, \quad \text{for any } y \in \bigl(F(x)\backslash\{y_0\}\bigr) - y_0 \text{ and } z \in G(x).$$

Thus, for any $x \in K$, we have

$$\sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}} \bigl\{f(y) - f(y_0) + g(z)\bigr\} > 0, \quad \text{for any } y \in F(x)\backslash\{y_0\} \text{ and } z \in G(x), \tag{6}$$

and, for any $x \in K$,

$$\sup_{f\in L}\bigl\{f(y) - f(y_0)\bigr\} + \sup_{g\in\Gamma}\bigl\{g(z)\bigr\} > 0, \quad \text{for any } y \in F(x)\backslash\{y_0\} \text{ and } z \in G(x). \tag{7}$$

It is clear that, for any $z \in G(x_0)$,

$$(0_V, z) \notin -\operatorname{int}(S \times P).$$

By Lemma 2.1(v),

$$\xi_{k0}(0_V, z) = \sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}} \left\{\frac{f(0_V)+g(z)}{f(k)+g(k)}\right\} \geqslant 0.$$

Namely, by (1),

$$\sup_{g\in\Gamma}\bigl\{g(z)\bigr\} \geqslant 0, \quad \text{for any } z \in G(x_0). \tag{8}$$

Due to $x_0 \in K$, there exists $b \in G(x_0)$ such that $-b \in P$.

Obviously,

$$(0_V, b) \in -\partial(S \times P).$$

By Lemma 2.1(iii),

$$\xi_{k0}(0_V, b) = \sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}} \left\{\frac{f(0_V)+g(b)}{f(k)+g(k)}\right\} = 0. \tag{9}$$

By (1), (8) and (9), we have

$$\sup_{g\in\Gamma}\bigl\{g(b)\bigr\} = 0. \tag{10}$$

Therefore, by (7), (8) and (10), we have

$$\inf\biggl(\bigcup_{x\in K}\biggl(\bigcup_{y\in F(x)}\sup_{f\in L}\bigl\{f(y) - f(y_0)\bigr\} + \bigcup_{z\in G(x)}\sup_{g\in\Gamma}\bigl\{g(z)\bigr\}\biggr)\biggr) = 0.$$

By (7) and (8), we get

$$\inf\left(\bigcup_{z\in G(x_0)}\sup_{g\in\Gamma}\{g(z)\}\right)=0,$$

and the proof is complete. □

Remark 3.1. By Corollary 2.1, there exist $L'\subset S^+$ and $\Gamma'\subset P^+$, which satisfy $f(k)+g(k)=1$, for any $f\in L'$, $g\in\Gamma'$ and $(f,g)\neq(0_{V^*},0_{Z^*})$, such that

$$S:=\{x\in V\mid f(x)\geqslant 0,\ \text{for } f\in L'\}$$

and

$$P:=\{z\in V\mid g(z)\geqslant 0,\ \text{for } g\in\Gamma'\}.$$

Therefore, there exist $L\subset S^+$ and $\Gamma\subset P^+$, which satisfy (1), (2) and (3).

Theorem 3.2. *Suppose that the following conditions are satisfied*:

(i) $x_0\in K$ *and* $y_0\in F(x_0)$;
(ii) *there exist* $L\subset S^+$ *and* $\Gamma\subset P^+$ *and* $L\times\Gamma\neq(0_{V^*},0_{Z^*})$, *such that*

$$\sup_{\substack{f\in L,\,g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}}\{f(y)-f(y_0)+g(z)\}>0,\quad \textit{for any } y\in F(x)\backslash\{y_0\} \textit{ and } z\in G(x). \tag{11}$$

Then, (x_0,y_0) *is a minimal solution of* (GVP).

Proof. Suppose that (x_0,y_0) is not a minimal solution of (GVP), then, there exists $x^*\in K$ such that

$$\left(F(x^*)-y_0\right)\cap\left(-S\backslash\{0_V\}\right)\neq\emptyset,$$

namely, there exists $y_1\in F(x^*)$ such that

$$y_1-y_0\in -S\backslash\{0_V\}.$$

Since $L\subset S^+$,

$$f(y_1)-f(y_0)\leqslant 0,\quad \text{for any } f\in L.$$

Namely,

$$\sup_{f\in L}\{f(y_1)-f(y_0)\}\leqslant 0. \tag{12}$$

Due to $x^*\in K$, this implies that there exists $b\in G(x^*)$ such that $-b\in P$. Thus,

$$g(b)\leqslant 0,\quad \text{for any } g\in\Gamma.$$

Namely,

$$\sup_{g\in\Gamma}\{g(b)\}\leqslant 0. \tag{13}$$

Adding (12) to (13), we have

$$\sup_{f\in L}\{f(y_1)-f(y_0)\}+\sup_{g\in\Gamma}\{g(b)\}\leqslant 0,$$

which contradicts (11). Hence, the proof is complete. □

Corollary 3.1. *Suppose that $x_0 \in K$ and $y_0 \in F(x_0)$. Then, (x_0, y_0) is a minimal solution of* (GVP) *if and only if there exist $L \subset S^+$ and $\Gamma \subset P^+$ and $L \times \Gamma \neq (0_{V^*}, 0_{Z^*})$, such that for any $x \in K$,*

$$\sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}}\{f(y)-f(y_0)+g(z)\}>0, \quad \textit{for any } y\in F(x)\backslash\{y_0\} \textit{ and } z\in G(x).$$

Proof. By (4) and Theorems 3.2, this corollary holds. □

Now we discuss vector program. Let $F : D \to V$ and $G : D \to Z$ be vector-valued functions. Thus, problem (GVP) becomes vector program:

$$\text{(VP)} \quad \min F(x),$$
$$\text{s.t.} \quad G(x)\in -P, \quad x\in D.$$

By Theorems 3.1 and 3.2 and Corollary 3.1, we have following results.

Corollary 3.2. *If $(x_0, F(x_0))$ is minimal solution of* (*VP*)*, then, for any $L \subset S^+$ and $\Gamma \subset P^+$, which satisfy*

$$f(k)+g(k)=1, \quad \textit{for } f\in L,\ g\in\Gamma \textit{ and } (f,g)\neq(0_{V^*},0_{Z^*}),$$
$$S:=\{x\in V \mid f(x)\geqslant 0,\ \textit{for any } f\in L\},$$

and

$$P:=\{z\in V \mid g(z)\geqslant 0,\ \textit{for any } g\in\Gamma\},$$

we have

$$\inf\left(\bigcup_{x\in K}\left(\sup_{f\in L}\{f(F(x))-f(F(x_0))\}+\sup_{g\in\Gamma}\{g(G(x))\}\right)\right)=0$$

and

$$\sup_{g\in\Gamma}\{g(G(x_0))\}=0.$$

Corollary 3.3. *There exist $L \subset S^+$ and $\Gamma \subset P^+$ and $L \times \Gamma \neq (0_{V^*}, 0_{Z^*})$, such that*

$$\sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}}\{f(F(x))-f(F(x_0))+g(G(x))\}>0,$$
$$\textit{for any } x\in K \textit{ and } F(x)\neq F(x_0).$$

Then, $(x_0, F(x_0))$ is a minimal solution of (VP).

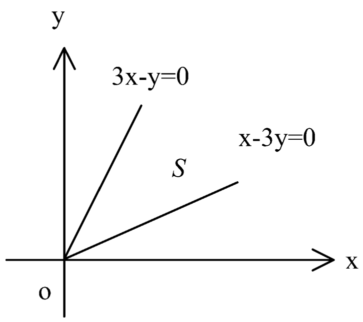

Fig. 2. The set S.

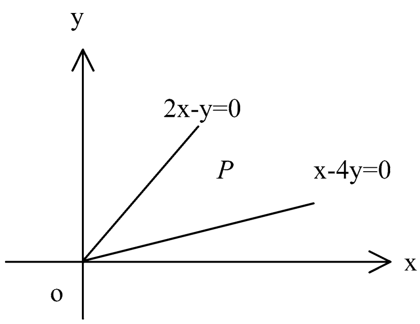

Fig. 3. The set P.

Corollary 3.4. $(x_0, F(x_0))$ *is a minimal solution of* (VP) *if and only if there exist* $L \subset S^+$ *and* $\Gamma \subset P^+$ *and* $L \times \Gamma \neq (0_{V^*}, 0_{Z^*})$*, such that, for any* $x \in K$ *and* $F(x) \neq F(x_0)$*,*

$$\sup_{\substack{f\in L,\, g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}} \big\{ f\big(F(x)\big) - f\big(F(x_0)\big) + g\big(G(x)\big)\big\} > 0.$$

Example 3.1. Let $V = R^2$ and $Z = R^2$. Given S and P (see Figs. 2 and 3, respectively). Suppose that $L = \{f_1, f_2\}$ and $\Gamma = \{g_1, g_2\}$, where

$$f_1(z) = (3, -1)\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad f_2(z) = (-1, 3)\begin{pmatrix} x \\ y \end{pmatrix},$$
$$g_1(z) = (2, -1)\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad g_2(z) = (-1, 4)\begin{pmatrix} x \\ y \end{pmatrix},$$
$$z = (x, y).$$

Then,

$$S := \big\{ z \in V \mid f(z) \geqslant 0, \text{ for any } f \in L \big\}$$
$$= \left\{ z \in R^2 \,\middle|\, f_1(z) = (3, -1)\begin{pmatrix} x \\ y \end{pmatrix} \geqslant 0,\ f_2(z) = (-1, 3)\begin{pmatrix} x \\ y \end{pmatrix} \geqslant 0 \right\}$$

and

$$P := \{z \in R^2 \mid g(z) \geqslant 0, \text{ for any } g \in \Gamma\}$$
$$= \left\{z \in R^2 \,\middle|\, g_1(z) = (2,-1)\begin{pmatrix} x \\ y \end{pmatrix} \geqslant 0,\ g_2(z) = (-1,4)\begin{pmatrix} x \\ y \end{pmatrix} \geqslant 0\right\}.$$

Consider vector program:

$$\text{(VP)} \quad \min F(x),$$
$$\text{s.t.} \quad G(x) \in -P, \quad x \in D,$$

where $F(x) = (F_1(x), F_2(x))$.

Thus, feasible set of (VP)

$$K = \{x \in D \mid G(x) \in -P\}.$$

Since $G(x) \in -P$, for any $x \in K$, $\sup_{g\in\Gamma}\{g(G(x))\} \leqslant 0$. Thus, by Corollary 3.4, $(x_0, F(x_0))$ is a minimal solution of (VP) if and only if for any $x \in K$ with $F(x) \neq F(x_0)$

$$\max Q(x) > 0, \tag{14}$$

where

$$Q(x) = \left\{(3,-1)\begin{pmatrix} F_1(x) - F_1(x_0) \\ F_2(x) - F_2(x_0) \end{pmatrix},\ (-1,3)\begin{pmatrix} F_1(x) - F_1(x_0) \\ F_2(x) - F_2(x_0) \end{pmatrix}\right\}.$$

Set

$$\mu(x) = 3F_1(x) - F_2(x), \qquad \nu(x) = -F_1(x) + 3F_2(x) \quad \text{and}$$
$$\Phi(x) = (\mu(x), \nu(x)).$$

We introduce multiobjective program problem:

$$\text{(MPP)} \quad \min \Phi(x), \quad x \in K.$$

If $(x_0, F(x_0))$ satisfies (14), it is clear that x_0 is a Pareto-optimal solution of (MPP). Conversely, if x_0 is a Pareto-optimal solution of (MPP), then, there exists no $x \in K$, such that

$$(\mu(x), \nu(x)) \leqslant (\mu(x_0), \nu(x_0)).$$

Thus, (14) holds.

Hence, it follows from (14) that $(x_0, F(x_0))$ is a minimal solution of (VP) if and only if x_0 is a Pareto-optimal solution of (MPP). This result shows that the solution of (VP) is reduced to the solution of (MPP).

Example 3.2. Let $V = R^2$, $X = R$ and $Z = R$. Given S (see Fig. 2), $P = R^+$ and $D = [-1, +\infty)$. Let

$$F(x) = \{(x, y) \mid -x \leqslant y \leqslant 1\} \quad \text{for any } x \in D,$$
$$G(x) = x - 1 \quad \text{for any } x \in D.$$

Consider generalized vector program problem:

$$\text{(GVP)}\quad \min F(x),$$
$$\text{s.t.}\quad -G(x)\cap P\neq\emptyset,\quad x\in D.$$

Thus, feasible set of (GVP)

$$K=\bigl\{x\in D\mid G(x)\in -R^+\bigr\}=[-1,1].$$

By Definition 3.1, minimal solution of (GVP) is equivalent to

$$N_1=\bigl\{\bigl(x,\bigl(-x,(1-x)/2\bigr)\bigr)\in K\times V\mid -1\leqslant x\leqslant 1\bigr\}.$$

On the other hand, for any $x\in K$, $z\in F(x)$ and $z\neq z_0$, where $z=(x,y)$ and $z_0=(x_0,y_0)$, we have

$$\sup_{\substack{f\in L,\,g\in\Gamma,\\ (f,g)\neq(0_{V^*},0_{Z^*})}}\bigl\{f(z)-f(z_0)+g\bigl(G(x)\bigr)\bigr\}$$
$$=\max\bigl\{3(x-x_0)-(y-y_0),3(y-y_0)-(x-x_0)\bigr\}.$$

Thus, by Corollary 3.1, (x_0,z_0) is a minimal solution of (GVP) if and only for any $x\in K$, $0\leqslant y\leqslant 1$ and $y\neq y_0$ we have

$$\max\bigl\{3(x-x_0)-(y-y_0),3(y-y_0)-(x-x_0)\bigr\}>0. \tag{15}$$

Set

$$\varphi(x,y)=3x-y,\qquad \phi(x,y)=3y-x\quad\text{and}$$
$$\Omega(x,y)=\bigl(\varphi(x,y),\phi(x,y)\bigr).$$

We introduce multiobjective programming problem:

$$\text{(MPP)}\quad \min\Omega(x,y),\quad -1\leqslant x\leqslant 1,\ -x\leqslant y\leqslant 1.$$

Thus, by (15), (x_0,z_0) is a minimal solution of (GVP) if and only if (x_0,y_0) is a weak Pareto-optimal solution of (MPP). By definition of weak Pareto-optimal solution, we can get the solution set of (MPP) is equivalent to

$$N_2=\bigl\{\bigl(x,\bigl(-x,(1-x)/2\bigr)\bigr)\in K\times V\mid -1\leqslant x\leqslant 1\bigr\}.$$

Clearly, $N_1=N_2$. Hence, this result shows that general cone-efficiency can reduce weak Pareto-efficiency.

For some nonvoid set $L\subset V^*$, let $R^L:=\prod_L R$ denote the product space in the product topology, which is a locally convex Hausdorff topological vector space (see Ref. [10]). Let $R^L_+:=\prod_L R^+$, where $R^+=\{r\in R\mid r\geqslant 0\}$. Suppose that

$$k\in\operatorname{int}S\quad\text{and}\quad e\in\operatorname{int}P,$$
$$f(k)+g(e)=1,\quad\text{for } f\in L,\ g\in\Gamma \text{ and } (f,g)\neq(0_{V^*},0_{Z^*}),$$
$$S:=\bigl\{z\in V\mid f(z)\geqslant 0,\ \text{for } f\in L\bigr\},$$

and

$$P:=\bigl\{z\in Z\mid g(z)\geqslant 0,\ \text{for } g\in\Gamma\bigr\}.$$

Consider vector program:

$$\begin{aligned}(\mathrm{VP})\quad & \min F(x),\\ & \text{s.t.}\quad G(x)\in -P,\quad x\in D.\end{aligned}$$

By Corollary 3.4, $(x_0, F(x_0))$ is a minimal solution of (VP) if and only if for any $x \in K$ and $F(x) \neq F(x_0)$,

$$\sup_{\substack{f\in L,\, g\in \Gamma,\\ (f,g)\neq(0_{Y^*},0_{Z^*})}} \bigl\{f\bigl(F(x)\bigr) - f\bigl(F(x_0)\bigr) + g\bigl(G(x)\bigr)\bigr\} > 0. \tag{16}$$

Since $G(x) \in -P$, for any $x \in K$, $\sup_{g\in\Gamma}\{g(G(x)\} \leqslant 0$.

Thus, (16) is equivalent to

$$\sup_{f\in L\setminus\{0_{Y^*}\}} \bigl\{f\bigl(F(x)\bigr) - f\bigl(F(x_0)\bigr)\bigr\} > 0, \quad \text{for any } x \in K \text{ and } F(x) \neq F(x_0). \tag{17}$$

We introduce generalized function $\Phi(x) = \prod_L f(F(x))$ and generalized multiobjective programming problem:

$$(\mathrm{GMPP})\quad \min \Phi(x),\quad x \in K.$$

A point $x_0 \in K$ is called a generalized Pareto-optimal solution of (GMPP) if there exists no $x \in K$ such that

$$\Phi(x_0) - \Phi(x) \in R_+^L \setminus \{0_{R^L}\}.$$

We have the following result.

Theorem 3.3. $(x_0, F(x_0))$ *is a minimal solution of* (VP) *if and only if* x_0 *is a generalized Pareto-optimal solution of* (GMPP).

Proof. If $(x_0, F(x_0))$ is a minimal solution of (VP), by Corollary 3.4 and (17), x_0 is a generalized Pareto-optimal solution of (GMPP).

Conversely, if x_0 is a generalized Pareto-optimal solution of (GMPP), there exists no $x \in K$, such that

$$\Phi(x_0) - \Phi(x) \in R_+^L \setminus \{0_{R^L}\}.$$

Thus, (17) holds. By Corollary 3.4, $(x_0, F(x_0))$ is a minimal solution of (VP). This completes the proof. □

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